The Natural Number System

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Abstract

The natural extension of $n!$ has finite values for negative integers and enables an upgrade of the factorial number system into the natural number system b–!. It also introduces relative binomial symbols.

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1 Introduction

The *factorial number system*, described on MathPages [1], is constructed by replacing powers 2^n or 10^n with $n!$. This system contains only whole numbers, so the primary question is whether it can be extended to represent all real numbers.

Standard definitions do not allow a meaningful value of $(-1)!$, and the gamma function has poles at $0, -1, -2, \ldots$. However, a suitable redefinition removes these singularities and provides a consistent extension of the factorial.

2 The Natural Number System

Direct computation of $(-1)!$ from the recurrence

$$(n+1)! = (n+1)n!$$

leads to contradictions, and $\Gamma(x)$ has poles at negative integers. Therefore, $(-1)!$ cannot serve as a base element of a number system.

2.1 Natural extension of the factorial

Several independent arguments lead to the same extension. One of them follows from symmetry properties of powers 2^n and factorials $n!$, another from differentiation and integration of polynomials, and a third from recursion.

Definition

The *natural extension of the factorial* is defined by

$$n!(1-n)! = 1, \qquad \forall n \in \mathbb{Z} \cup \{0\}. \tag{1}$$

We write $(m)!$ to emphasize that m may be any integer. For $n \geq 0$, $(n)! = n!$, and for $n > 0$,

$$(-n)! = \frac{1}{(n+1)!}.$$

An equivalent recursive definition is

$$0! = 1, \qquad (m+1)! = (m+1)(m)! \quad \forall m \in \mathbb{Z}. \tag{2}$$

2.2 Factorial decomposition

Every integer $n!$ admits a finite decomposition

$$n! = \sum_{k=0}^{n-1} c_k \, k!, \qquad 0 \le c_k \le k, \tag{3}$$

and this decomposition extends to negative indices as an infinite sum:

$$1 = \sum_{k=1}^{\infty} (-k)!. \tag{4}$$

Thus every real number $x \ge 0$ has a unique representation

$$x = \sum_{k=-\infty}^{m} c_k (k)!, \tag{5}$$

where the coefficients $c_k \in \mathbb{Z}_{\ge 0}$ satisfy $0 \le c_k \le |k|$. The separator is placed at $0!$.

2.3 Examples

$$\frac{1}{2} = 0,1, \qquad \frac{1}{3} = 0,02, \qquad \frac{1}{4} = 0,012$$

$$e = 10,11111\ldots$$

$$\cosh(1) = 1,1010\ldots, \qquad \sinh(1) = 1,01\ldots$$

In contrast to decimal notation, every rational number has a finite representation in the natural number system.

2.4 Arithmetic

Addition is performed digitwise with carrying:

$$c_k(x+y) = c_k(x) + c_k(y) \pmod{k+1}, \tag{6}$$

where a carry of 1 is propagated to the next factorial place if the sum exceeds k.

Multiplication cannot be reduced to digitwise operations, because in general

$$m!n! \neq (m+n)!,$$

except in special cases such as $6! \cdot 7! = 10!$. The obstruction is given by the binomial coefficient.

3 Relative Binomial Symbols

Classical binomial coefficients are extended using the natural factorial:

$$\binom{m}{l} = \frac{(m)!}{(l)!(m-l)!}, \qquad m, l \in \mathbb{Z}. \tag{7}$$

Definition

For integers a, b, c, d we define the *relative binomial symbol* by

$$\binom{a-b}{c-d} = \frac{(a-b)!}{(c-d)!(a-b-c+d)!}. \tag{8}$$

For nonnegative arguments this reduces to the classical binomial coefficient. For negative arguments it yields transformations of ordinary binomial coefficients and coincides with coefficients counting combinations with repetition.

3.1 Connections

Relative binomial coefficients appear naturally in

- N-nomial coefficients,
- Bose–Einstein statistics,
- combinatorics with repetition,
- rotations of Pascal-type triangles.

Eigen-diagonals of these structures reproduce Fibonacci numbers, and limiting cases connect to the golden ratio φ.

4 Mirror symmetry

Define the *mirror map*

$$f(k) = -k.$$

For

$$x = \sum_{k=-\infty}^{\infty} c_k (k)!$$

its mirror is

$$\tilde{x} = \sum_{k=-\infty}^{\infty} c_k (-k)!.$$

Natural numbers and rational numbers from $(0, 1)$ form mirror sets in the b–! system. Integers correspond to left-infinite representations, fractions to right-infinite ones.

5 Physical interpretation

The natural number system appears implicitly in Planck's radiation law, where the factor

$$\frac{x}{e^x - 1}$$

renormalizes the classical harmonic oscillator energy. The series expansion of e^x at $x = 1$ is homomorphic to the additive group underlying the natural number system.

Further connections arise with

- the fine-structure constant,
- the quantum Hall effect,
- Feigenbaum's constant,
- Gaussian distributions.

6 Conclusion

The natural extension of the factorial produces a complete number system in which all rational numbers have finite representations, irrational numbers are infinite, and relative binomial symbols arise naturally. This framework unifies combinatorics, number systems, and several structures occurring in physics.

References

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